New characterizations of simple points, minimal non-simple sets and P-simple points in 2D, 3D and 4D discrete spaces

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Abstract. In this article, we present new results on simple points, minimal non-simple sets (MNS) and P-simple points. In particular, we propose new characterizations which hold in dimensions 2, 3 and 4, and which lead to efficient algorithms for detecting such points or sets. This work is settled in the framework of cubical complexes, and some of the main results are based on the properties of critical kernels.

Introduction

Topology-preserving operators, like homotopic skeletonization, are used in many applications of image analysis to transform an object while leaving unchanged its topological characteristics. In discrete grids ($\mathbb{Z}^2$, $\mathbb{Z}^3$, $\mathbb{Z}^4$), such a transformation can be defined thanks to the notion of simple point [20]: intuitively, a point of an object is called simple if it can be deleted from this object without altering topology.

The most "natural" way to thin an object consists in removing some of its border points in parallel, in a symmetrical manner. However, parallel deletion of simple points does not guarantee topology preservation in general. In fact, such a guarantee is not obvious to obtain, even for the 2D case (see [10]). C. Ronse introduced the minimal non-simple sets [28] to study the conditions under which points may be removed simultaneously while preserving topology of 2D objects. This leads to verification methods for the topological soundness of parallel thinning algorithms. Such methods have been proposed for 2D algorithms by C. Ronse [28] and R. Hall [15], they have been developed for the 3D case by T.Y. Kong [21, 16] and C.M. Ma [25], as well as for the 4D case by C-J. Gau and T.Y. Kong [13, 19]. For the 3D case, G. Bertrand [1] introduced the notion of P-simple point as a verification method but also as a methodology to design parallel thinning algorithms [2, 9, 23, 24].

Introduced recently by G. Bertrand, critical kernels [3, 4] constitute a general framework settled in the category of abstract complexes for the study of parallel thinning in any dimension. It allows easy design of parallel thinning algorithms which produce new types of skeletons, with specific geometrical properties, while guaranteeing their topological soundness [7, 5, 6]. A new definition of a simple point is proposed in [3, 4], based on the collapse operation which is a classical tool in algebraic topology and which guarantees topology preservation. Then, the notions of an *essential face* and of a *core* of a face allow to define the *critical kernel* $\mathcal{K}$ of a complex X. The most fundamental result proved in [3, 4] is that, if a subset Y of X contains $\mathcal{K}$, then X collapses onto Y, hence X and Y "have the same topology".

In this article, we present new results on simple points, minimal non-simple sets (MNS), P-simple points and critical kernels. Let us summarize the main ones among these results.

First of all, we state some *confluence properties* of the collapse operation (Th. 5, Th. 6), which play a fundamental role in the proof of forthcoming theorems. These properties do not hold in general due to the existence of "topological monsters" such as Bing's house ([8], see also [27]); we show that they are indeed true in some discrete spaces which are not large enough to contain such counter-examples.

Based on these confluence properties, we derive a new characterization of 2D, 3D and 4D simple points (Th. 7) which leads to a simple, greedy linear time algorithm for simplicity checking.

Then, we show the equivalence (up to 4D) between the notion of MNS and the notion of crucial clique, derived from the framework of critical kernels. This equivalence (Th. 21) leads to the first characterization of MNS which can be verified using a polynomial method.

Finally, we show the equivalence between the notion of P-simple point and the notion of weakly crucial point, also derived from the framework of critical kernels. This equivalence (Th. 23) leads to the first local characterization of P-simple points in 4D.

This paper is self-contained, however the proofs cannot be included due to space limitation. They can be found in [12, 11], together with some illustrations and developments.

1 Cubical Complexes

Abstract complexes have been promoted in particular by V. Kovalevsky [22] in order to provide a sound topological basis for image analysis. For instance, in this framework, we retrieve the main notions and results of digital topology, such as the notion of simple point.

Intuitively, a cubical complex may be thought of as a set of elements having various dimensions (*e.g.* cubes, squares, edges, vertices) glued together according to certain rules. In this section, we recall briefly some basic definitions on complexes, see also [7, 5, 6] for more details. We consider here n-dimensional complexes, with $0 \leq n \leq 4$.

Let S be a set. If T is a subset of S, we write $T \subseteq S$. We denote by $|S|$ the number of elements of S.

Let $\mathbb{Z}$ be the set of integers. We consider the families of sets $\mathbb{F}^1_0$, $\mathbb{F}^1_1$, such that $\mathbb{F}^1_0 = \{\{a\} \mid a \in \mathbb{Z}\}$, $\mathbb{F}^1_1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. A subset f of $\mathbb{Z}^n$, $n \geq 2$, which is the Cartesian product of exactly m elements of $\mathbb{F}^1_1$ and $(n-m)$ elements of

$\mathbb{F}_0^1$ is called a *face* or an *m-face* of $\mathbb{Z}^n$, m is the *dimension of f*, we write $\dim(f) = m$.

Observe that any non-empty intersection of faces is a face. For example, the intersection of two 2-faces A and B may be either a 2-face (if $A = B$), a 1-face, a 0-face, or the empty set.

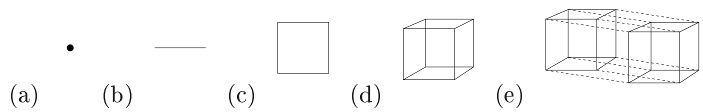

Fig. 1. Graphical representations of: (a) a 0-face, (b) a 1-face, (c) a 2-face, (d) a 3-face, (e) a 4-face.

We denote by $\mathbb{F}^n$ the set composed of all m-faces of $\mathbb{Z}^n$, with $0 \leq m \leq n$. An m-face of $\mathbb{Z}^n$ is called a *point* if $m = 0$, a *(unit) interval* if $m = 1$, a *(unit) square* if $m = 2$, a *(unit) cube* if $m = 3$, a *(unit) hypercube* if $m = 4$ (see Fig. 1).

Let f be a face in $\mathbb{F}^n$. We set $\hat{f} = \{g \in \mathbb{F}^n \mid g \subseteq f\}$ and $\hat{f}^* = \hat{f} \setminus \{f\}$.
Any $g \in \hat{f}$ is a *face of f*, and any $g \in \hat{f}^*$ is a *proper face of f*.
If X is a finite set of faces in $\mathbb{F}^n$, we write $X^- = \cup\{\hat{f} \mid f \in X\}$, X^- is the *closure of X*.

A set X of faces in $\mathbb{F}^n$ is a *cell* or an *m-cell* if there exists an m-face $f \in X$, such that $X = \hat{f}$. The *boundary of a cell $\hat{f}$* is the set $\hat{f}^*$.

A finite set X of faces in $\mathbb{F}^n$ is a *complex (in $\mathbb{F}^n$)* if $X = X^-$. Any subset Y of a complex X which is also a complex is a *subcomplex of X*. If Y is a subcomplex of X, we write $Y \preceq X$. If X is a complex in $\mathbb{F}^n$, we also write $X \preceq \mathbb{F}^n$. In Fig. 2 and Fig. 3, some complexes are represented. Notice that any cell is a complex.

Let $X \subseteq \mathbb{F}^n$ be a non-empty set of faces. A sequence $(f_i)_{i=0}^{\ell}$ of faces of X is a *path in X (from f_0 to f_ℓ)* if either f_i is a face of f_{i+1} or f_{i+1} is a face of f_i, for all $i \in [0, \ell - 1]$. We say that X *is connected* if, for any two faces f, g in X, there is a path from f to g in X; otherwise we say that X *is disconnected*. We say that Y is a *connected component of* X if $Y \subseteq X$, Y is connected and if Y is maximal for these two properties (*i.e.*, we have $Z = Y$ whenever $Y \subseteq Z \subseteq X$ and Z is connected).

Let $X \subseteq \mathbb{F}^n$. A face $f \in X$ is *a facet of X* if there is no $g \in X$ such that $f \in \hat{g}^*$. We denote by X^+ the set composed of all facets of X.
If X is a complex, observe that in general, X^+ is not a complex, and that $[X^+]^- = X$.

Let $X \preceq \mathbb{F}^n$, $X \neq \emptyset$, the number $\dim(X) = \max\{\dim(f) \mid f \in X^+\}$ is the *dimension of* X. We say that X is an m-complex if $\dim(X) = m$.
We say that X is *pure* if, for each $f \in X^+$, we have $\dim(f) = \dim(X)$.
In Fig. 2, the complexes (a) and (f) are pure, while (b,c,d,e) are not.

2 Collapse and simple sets

Intuitively a subcomplex of a complex X is simple if its removal from X "does not change the topology of X". In this section we recall a definition of a simple subcomplex based on the operation of collapse [14], which is a discrete analogue of a continuous deformation (a homotopy).

Let X be a complex in $\mathbb{F}^n$ and let $f \in X$. If there exists one face $g \in \hat{f}^*$ such that f is the only face of X which strictly includes g, then g is said to be *free for* X and the pair (f, g) is said to be a *free pair for* X. Notice that, if (f, g) is a free pair, then we have necessarily $f \in X^+$ and $\dim(g) = \dim(f) - 1$.

Let X be a complex, and let (f, g) be a free pair for X. The complex $X \setminus \{f, g\}$ is an *elementary collapse of* X.

Let X, Y be two complexes. We say that X *collapses onto* Y if $Y = X$ or if there exists a *collapse sequence from* X *to* Y, *i.e.*, a sequence of complexes $\langle X_0, ..., X_\ell \rangle$ such that $X_0 = X$, $X_\ell = Y$, and X_i is an elementary collapse of X_{i-1}, $i = 1, ..., \ell$. If X collapses onto Y and Y is a complex made of a single point, we say that *collapses onto a point*.

Fig. 2 illustrates a collapse sequence. Observe that, if X is a cell of any dimension, then X collapses onto a point. It may easily be seen that the collapse operation preserves the number of connected components.

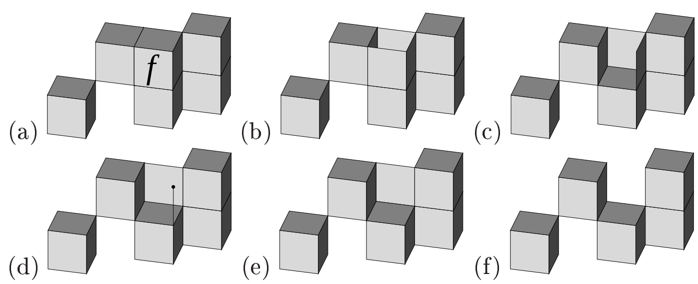

Fig. 2. (a): a pure 3-complex $X \preceq \mathbb{F}^3$, and a 3-face $f \in X^+$. (f): a complex Y which is the detachment of $\hat{f}$ from X. (a-f): a collapse sequence from X to Y.

Let X, Y be two complexes. Let Z such that $X \cap Y \preceq Z \preceq Y$, and let $f, g \in Z \setminus X$. It may be seen that the pair (f, g) is a free pair for $X \cup Z$ if and only if (f, g) is a free pair for Z. Thus, by induction, we have the following property.

Proposition 1 ([3, 4])**.** *Let $X, Y \preceq \mathbb{F}^n$. The complex $X \cup Y$ collapses onto X if and only if Y collapses onto $X \cap Y$.*

The operation of detachment allows to remove a subset from a complex, while guaranteeing that the result is still a complex.

Definition 2 ([3, 4])**.** *Let $Y \subseteq X \preceq \mathbb{F}^n$. We set $X \oslash Y = (X^+ \setminus Y^+)^-$. The set $X \oslash Y$ is a complex which is the* detachment *of Y from X.*

In the following, we will be interested in the case where Y is a single cell. For example in Fig. 2, we see a complex X (a) containing a 3-cell $\hat{f}$, and $X \oslash \hat{f}$ is depicted in (f).

Let us now recall here a definition of simplicity based on the collapse operation, which can be seen as a discrete counterpart of the one given by T.Y. Kong [17].

Definition 3 ([3,4])**.** *Let $Y \subseteq X$; we say that Y is* simple for X *if X collapses onto $X \oslash Y$.*

The collapse sequence displayed in Fig. 2 (a-f) shows that the cell $\hat{f}$ is simple for the complex depicted in (a).

The notion of attachment, as introduced by T.Y. Kong [16,17], leads to a local characterization of simple sets, which follows easily from Prop. 1.

Let $Y \preceq X \preceq \mathbb{F}^n$. The *attachment* of Y for X is the complex defined by $Att(Y, X) = Y \cap (X \oslash Y)$.

Proposition 4 ([3,4])**.** *Let $Y \preceq X \preceq \mathbb{F}^n$. The complex Y is simple for X if and only if Y collapses onto $Att(Y, X)$.*

Let us introduce informally the *Schlegel diagrams* as a graphical representation for visualizing the attachment of a cell. In Fig. 3a, the boundary of a 3-cell $\hat{f}$ and its Schlegel diagram are depicted. The interest of this representation lies in the fact that a structure like $\hat{f}^*$ lying in the 3D space may be represented in the 2D plane. Notice that one 2-face of the boundary, here the square $efhg$, is not represented by a closed polygon in the schlegel diagram, but we may consider that it is represented by the outside space.

As an illustration of Prop. 4, Fig. 3b shows (both directly and by its Schlegel diagram) the attachment of $\hat{f}$ for the complex X of Fig. 2a, and we can easily verify that $\hat{f}$ collapses onto $Att(\hat{f}, X)$.

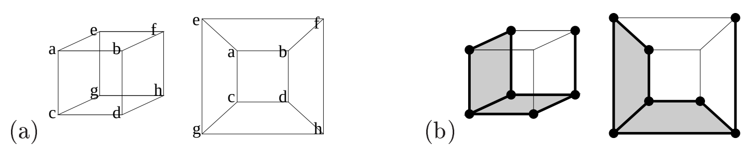

Fig. 3. (a): The boundary of a 3-cell and its Schlegel diagram. (b): The attachment of $\hat{f}$ for X (see Fig. 2a).

Representing 4D objects is not easy. To start with, let us consider Fig. 4a where a representation of the 3D complex X of Fig. 2a is given under the form of two horizontal cross-sections, each black dot representing a 3-cell.

In a similar way, we may represent a 4D object by its "3D sections", as the object Y in Fig. 4b. Such an object may be thought of as a "time series of 3D objects". In Fig. 4b, each black dot represents a 4-cell of the whole 4D complex Y.

Schlegel diagrams are particularly useful for representing the attachment of a 4D cell $\hat{f}$, whenever this attachment if not equal to $\hat{f}^*$. Fig. 5a shows the

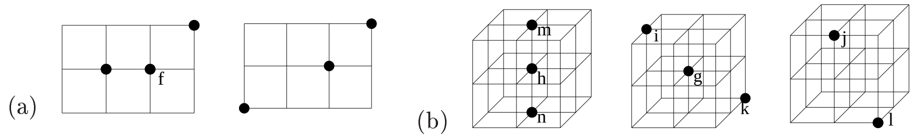

Fig. 4. (a): An alternative representation of the 3D complex X of Fig. 2a. (b): A similar representation of a 4D complex Y.

Schlegel diagram of the boundary of a 4-cell (see Fig. 1e), where one of the 3-faces is represented by the outside space. Fig. 5b shows the Schlegel diagram of the attachment of the 4-cell g in Y (see Fig. 4b). For example, the 3-cell H represented in the center of the diagram is the intersection between the 4-cell g and the 4-cell h. Also, the 2-cell I (resp. the 1-cell J, the 1-cell K, the 0-cell L) is $g \cap i$ (resp. $g \cap j$, $g \cap k$, $g \cap l$). The two 2-cells which are the intersections of g with, respectively, m and n, are both included in the 3-cell H. Observe that the cell g is not simple (its attachment is not connected).

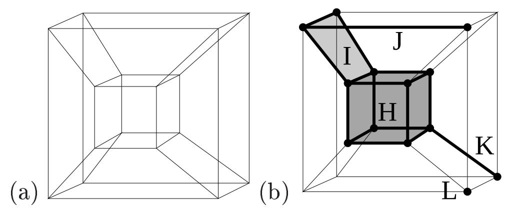

Fig. 5. (a): The Schlegel diagram of the boundary of a 4-cell. (b): The Schlegel diagram of the attachment of the 4-cell g of Fig. 4b, which is not simple.

3 Confluences

Let $X \preceq \mathbb{F}^n$. If f is a facet of X, then by Def. 3, $\hat{f}$ is simple if and only if X collapses onto $X \oslash \hat{f}$. From Prop. 4, we see that checking the simplicity of a cell f reduces to the search for a collapse sequence from $\hat{f}$ to $Att(\hat{f}, X)$. We will show in Sec. 4 that the huge number (especially in 4D) of possible such collapse sequences need not be exhaustively explored, thanks to the confluence properties (Th. 5 and Th. 6) introduced in this section.

Consider three complexes A, B, C. If A collapses onto C and A collapses onto B, then we know that A, B and C "have the same topology". If furthermore we have $C \preceq B \preceq A$, it is tempting to conjecture that B collapses onto C.

In the two-dimensional discrete plane $\mathbb{F}^2$, the above conjecture is true, we call it a confluence property. But quite surprisingly it does not hold in $\mathbb{F}^3$ (more generally in $\mathbb{F}^n, n \geq 3$), and this fact constitutes indeed one of the principal difficulties when dealing with certain topological properties, such as the Poincaré conjecture for example. A classical counter-example to this assertion is Bing's house ([8], see also [27]).

In the boundary of an n-face with $n \leq 4$, there is "not enough room" to build such counter-examples, and thus some kinds of confluence properties hold.

Theorem 5 (Confluence 1). *Let f be a d-face with $d \in \{2,3,4\}$, let $A, B \preceq \hat{f}^*$ such that $B \preceq A$, and A collapses onto a point. Then, B collapses onto a point if and only if A collapses onto B.*

The second confluence theorem may be easily derived from Th. 5 and the fact that $\hat{f}$ collapses onto a point.

Theorem 6 (Confluence 2). *Let f be a d-face with $d \in \{2,3,4\}$, and let $C, D \preceq \hat{f}^*$ such that $D \preceq C$, and $\hat{f}$ collapses onto D. Then, $\hat{f}$ collapses onto C if and only if C collapses onto D.*

4 New characterization of simple cells

In the image processing literature, a (binary) digital image is often considered as a set of pixels in 2D or voxels in 3D. A pixel is an elementary square and a voxel is an elementary cube, thus an easy correspondence can be made between this classical view and the framework of cubical complexes.

If $X \preceq \mathbb{F}^n$ and if X is a pure n-complex, then we write $X \sqsubseteq \mathbb{F}^n$. In other words, $X \sqsubseteq \mathbb{F}^n$ means that X^+ is a set composed of n-faces (*e.g.*, pixels in 2D or voxels in 3D). If $X, Y \sqsubseteq \mathbb{F}^n$ and $Y \preceq X$, then we write $Y \sqsubseteq X$.

Notice that, if $X \sqsubseteq \mathbb{F}^n$ and if $\hat{f}$ is an n-cell of X, then $X \oslash \hat{f} \sqsubseteq \mathbb{F}^n$. There is indeed an equivalence between the operation on complexes which consists of removing (by detachment) a simple n-cell, and the removal of an 8-simple (resp. 26-simple, 80-simple) point in the framework of 2D (resp. 3D, 4D) digital topology (see [16, 17, 7, 6]).

From Prop. 4 and Th. 6, we have the following characterization of a simple cell, which does only depend on the status of the faces which are in the cell.

Theorem 7. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2,3,4\}$. Let f be a facet of X, and let $A = Att(\hat{f}, X)$. The two following statements hold:*
i) The cell $\hat{f}$ is simple for X if and only if $\hat{f}$ collapses onto A.
ii) Suppose that $\hat{f}$ is simple for X. For any Z such that $A \preceq Z \preceq \hat{f}$, if $\hat{f}$ collapses onto Z then Z collapses onto A.

Now, thanks to Th. 7, if we want to check whether a cell $\hat{f}$ is simple or not, it is sufficient to apply the following greedy algorithm:
Set $Z = \hat{f}$;
Select any free pair (f, g) in $Z \setminus A$, and set Z to $Z \setminus \{f, g\}$;
Continue until either $Z = A$ (answer yes) or no such pair is found (answer no).

If this algorithm returns "yes", then obviously $\hat{f}$ collapses onto A and by i), $\hat{f}$ is simple. In the other case, we have found a subcomplex Z of A such that $\hat{f}$ collapses onto Z and Z does not collapse onto A, thus by the negation of ii), $\hat{f}$ is not simple.

This algorithm may be implemented to run in linear time with respect to the number of elements in the attachment of a cell.

5 Critical kernels

Let us briefly recall the framework introduced by one of the authors (in [3, 4]) for thinning, in parallel, discrete objects with the warranty that we do not alter the topology of these objects. We focus here on the two-, three- and four-dimensional cases, but in fact some of the results in this section are valid for complexes of arbitrary dimension. This framework is based solely on three notions: the notion of an essential face which allows us to define the core of a face, and the notion of a critical face.

Definition 8. *Let $X \preceq \mathbb{F}^n$ and let $f \in X$. We say that f is an* essential face for X *if f is precisely the intersection of all facets of X which contain f,* i.e.*, if $f = \cap\{g \in X^+ \mid f \subseteq g\}$. We denote by $Ess(X)$ the set composed of all essential faces of X. If f is an essential face for X, we say that $\hat{f}$ is an* essential cell for X*. If $Y \preceq X$ and $Ess(Y) \subseteq Ess(X)$, then we write $Y \trianglelefteq X$.*

Observe that a facet of X is necessarily an essential face for X, *i.e.*, $X^+ \subseteq Ess(X)$. Observe also that, if X and Y are both pure n-complexes, we have that $Y \trianglelefteq X$ whenever Y is a subcomplex of X.

Definition 9. *Let $X \preceq \mathbb{F}^n$ and let $f \in Ess(X)$. The* core of $\hat{f}$ for X *is the complex $Core(\hat{f}, X) = \cup\{\hat{g} \mid g \in Ess(X) \cap \hat{f}^*\}$.*

Proposition 10 ([3])**.** *Let $X \preceq \mathbb{F}^n$, and let $f \in Ess(X)$. Let $K = \{g \in X \mid f \subseteq g\}$, and let $Y = X \oslash K$. We have: $Core(\hat{f}, X) = Att(\hat{f}, Y \cup \hat{f}) = \hat{f} \cap Y$.*

Corollary 11 ([3, 4])**.** *Let $X \preceq \mathbb{F}^n$, and let $f \in X^+$. We have: $Core(\hat{f}, X) = Att(\hat{f}, X)$.*

Definition 12. *Let $X \preceq \mathbb{F}^n$ and let $f \in X$. We say that f and $\hat{f}$ are* regular for X *if $f \in Ess(X)$ and if $\hat{f}$ collapses onto $Core(\hat{f}, X)$. We say that f and $\hat{f}$ are* critical *for X if $f \in Ess(X)$ and if f is not regular for X.*
If $X \preceq \mathbb{F}^n$, we set $Critic(X) = \cup\{\hat{f} \mid f$ is critical for $X\}$, we say that $Critic(X)$ is the critical kernel of X*.*
A face f in X is a maximal critical face*, or an* M-critical face (for X)*, if f is a facet of $Critic(X)$.*

In other words, f is an M-critical face if it is critical and not included in any other critical face.

Proposition 13 ([3, 4])**.** *Let $X \preceq \mathbb{F}^n$, and let $f \in Ess(X)$. Let $Y = \cup\{\hat{g} \mid g \in X^+$ and $f \subseteq g\}$ and $Z = [X \oslash Y] \cup \hat{f}$. The face f is regular for X if and only if $\hat{f}$ is simple for Z.*

The following theorem is the most fundamental result concerning critical kernels. We use it for the proofs of our main properties in dimension 4 or less, but notice that the theorem holds whatever the dimension.

Theorem 14 ([3, 4])**.** *Let $n \in \mathbb{N}$, let $X \preceq \mathbb{F}^n$.*
i) The complex X collapses onto its critical kernel.

ii) If $Y \trianglelefteq X$ contains the critical kernel of X, then X collapses onto Y.
iii) If $Y \trianglelefteq X$ contains the critical kernel of X, then any Z such that $Y \preceq Z \trianglelefteq X$ collapses onto Y.

If X is a pure n-complex (*e.g.*, a set of 3-cells, or voxels, in $\mathbb{F}^3$), the critical kernel of X is not necessarily a pure n-complex. The notion of crucial clique, introduced in [7], allows us to recover a pure n-subcomplex Y of an arbitrary pure n-complex X, under the constraint that X collapses onto Y.

Definition 15 ([7]). *Let $X \sqsubseteq \mathbb{F}^n$, and let f be an M-critical face for X. The set K of all the facets of X which contain f is called a* crucial clique (for X). *More precisely, K is the* crucial clique induced by f.

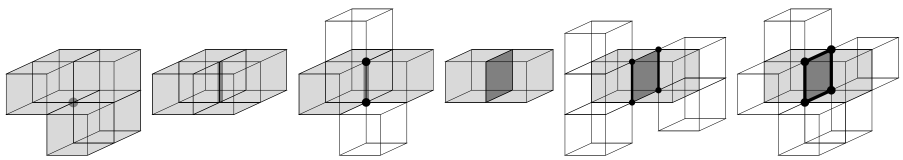
(a) (b) (c) (d) (e) (f)

Fig. 6. Crucial cliques in $\mathbb{F}^3$ (represented in light gray): (a) induced by an M-critical 0-face; (b,c) induced by an M-critical 1-face; (d,e,f) induced by an M-critical 2-face. The considered M-critical faces are in dark gray, the core of these M-critical faces (when non-empty) is represented in black.

Some 3D crucial cliques are illustrated in Fig. 6. By Th. 14 and the above definition, if a subcomplex $Y \sqsubseteq X \sqsubseteq \mathbb{F}^n$ contains all the critical facets of X, and at least one facet of each crucial clique for X, then X collapses onto Y.

Now, let us state two properties of crucial cliques which are essential for the proof of one of our main results (Th. 21).

Proposition 16. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2, 3, 4\}$, let f be an M-critical face of X, let K be the crucial clique induced by f, and let k be any facet of K. Let K' be such that $K' \subseteq K \setminus \{k\}$ and $K' \neq K \setminus \{k\}$.*
Then, k is a simple face of the complex $[X \oslash K']$.

Proposition 17. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2, 3, 4\}$, let f be an M-critical face of X, let K be the crucial clique induced by f, and let k be any facet of K.*
Then, k is not a simple face of the complex $[X \oslash K] \cup \hat{k}$.

6 Minimal non-simple sets

C. Ronse introduced in [28] the minimal non-simple sets (MNS) to propose some conditions under which simple points can be removed in parallel while preserving topology. This leads to verification methods for the topological soundness of 2D thinning algorithms [28, 15], 3D thinning algorithms [21, 16, 25], the 4D case has even been considered in [13, 18, 19].

The main result of this section (Th. 21) states the equivalence between MNS and crucial cliques in dimensions 2, 3 and 4. This equivalence leads to the first characterization of MNS which can be verified using a polynomial method. In contrast, the very definition of a MNS (see below), as well as the characterization of Th. 18, involves the examination of all subsets of a given candidate set, *e.g.*, a subset of a $2 \times 2 \times 2 \times 2$ block in 4D.

Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2, 3, 4\}$. A sequence $\langle k_0, \ldots, k_\ell \rangle$ of facets of X is said to be a *simple sequence for* X if k_0 is simple for X, and if, for any $i \in \{1, \ldots, \ell\}$, k_i is simple for $X \oslash \{k_j \mid 0 \leq j < i\}$. Let K be a set of facets of X. The set K is said to be *F-simple* (where "F" stands for facet) for X if K is empty, or if the elements of K can be ordered as a simple sequence for X. The set K is *minimal non-simple for* X if it is not F-simple for X and if all its proper subsets are F-simple. The following characterization will be used in the sequel.

Theorem 18 (adapted from Gau and Kong [13], theorem 3)**.** *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2, 3, 4\}$, and let $K \subseteq X^+$. Then K is a minimal non-simple set for X if and only if the two following conditions hold:*
i) Each k of K is non-simple for $[X \oslash K] \cup \hat{k}$.
ii) Each k of K is simple for $[X \oslash K']$ whenever $K' \subseteq K \setminus \{k\}$ and $K' \neq K \setminus \{k\}$.

For example, it may be seen that the sets displayed in Fig. 6 in light gray are indeed minimal non-simple sets.

Th. 19 is a key property[1] which is used to prove Prop. 20 and Th. 23.

Theorem 19. *Let f be a d-face with $d \in \{2, 3, 4\}$, let ℓ be an integer strictly greater than 1, let $X_1, \ldots, X_\ell$ be ℓ subcomplexes of $\hat{f}$. The two following assertions are equivalent:*
i) For all $L \subseteq \{1, \ldots, \ell\}$ such that $L \neq \emptyset$, $\cup_{i \in L} X_i$ collapses onto a point.
ii) For all $L \subseteq \{1, \ldots, \ell\}$ such that $L \neq \emptyset$, $\cap_{i \in L} X_i$ collapses onto a point.

Let us now establish the link between MNS and crucial cliques.

Proposition 20. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2, 3, 4\}$, let K be a minimal non-simple set for X, and let f be the intersection of all the elements of K. Then, f is an M-critical face for X and K is the induced crucial clique.*

If K is a crucial clique for X, then from Th. 18, Prop. 16 and Prop. 17, K is a minimal non-simple set for X. Conversely, if K is a minimal non-simple set for X, then by Prop. 20, K is a crucial clique. Thus, we have the following theorem.

Theorem 21. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2, 3, 4\}$, and let $K \subseteq X^+$. Then K is a minimal non-simple set for X if and only if it is a crucial clique for X.*

[1] Notice that a similar property holds in $\mathbb{R}^3$, in the framework of algebraic topology, if we replace the notion of collapsibility onto a point by the one of contractibility [18, 26].

7 P-simple points

In the preceding section, we saw that critical kernels which are settled in the framework of abstract complexes allow to derive the notion of a minimal non-simple set proposed in the context of digital topology. Also in the framework of digital topology, one of the authors introduced the notion of P-simple point [2], and proved for the 3D case a local characterization which leads to a linear algorithm for testing P-simplicity. In [7], we stated the equivalence between the notion of 2D P-simple points and a notion derived from the one of crucial clique. Here, we extend this equivalence result up to 4D.

Let $X \sqsubseteq \mathbb{F}^n$, and let $C \subseteq X^+$. A facet $k \in C$ is said to be *P-simple for* $\langle X, C\rangle$ if k is simple for all complexes $X \oslash T$, such that $T \subseteq C \setminus \{k\}$.

Definition 22. *Let $X \sqsubseteq \mathbb{F}^n$, and let C be a set of facets of X, we set $D = X^+ \setminus C$. Let $k \in C$, we say that k is* weakly crucial for $\langle X, D\rangle$ *if k contains a face f which is critical for X, and such that all the facets of X containing f are in C.*

Theorem 23. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2, 3, 4\}$, let C be a set of facets of X, let $D = X \oslash C$. Let $k \in C$, the facet k is P-simple for $\langle X, C\rangle$ if and only if k is not weakly crucial for $\langle X, D\rangle$.*

Conclusion

We provided in this article a new characterization of simple points, in dimensions up to 4D, leading to an efficient simplicity testing algorithm. Moreover, we demonstrated that the main concepts previously introduced in order to study topology-preserving parallel thinning in the framework of digital topology, namely P-simple points and minimal non-simple sets, may be not only retrieved in the framework of critical kernels, but also better understood and enriched. Critical kernels thus appear to constitute a unifying framework which encompasses previous works on parallel thinning.

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